

# THE $A$ -POLYNOMIAL OF THE $(-2, 3, 3 + 2n)$ PRETZEL KNOTS

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**ABSTRACT.** We show that the  $A$ -polynomial  $A_n$  of the 1-parameter family of pretzel knots  $K_n = (-2, 3, 3 + 2n)$  satisfies a linear recursion relation of order 4 with explicit constant coefficients and initial conditions. Our proof combines results of Tamura-Yokota and the second author. As a corollary, we show that the  $A$ -polynomial of  $K_n$  and the mirror of  $K_{-n}$  are related by an explicit  $\mathrm{GL}(2, \mathbb{Z})$  action. We leave open the question of whether or not this action lifts to the quantum level.

## 1. INTRODUCTION

**1.1. The behavior of the  $A$ -polynomial under filling.** In [CCG<sup>+</sup>94], the authors introduced the  $A$ -polynomial  $A_W$  of a hyperbolic 3-manifold  $W$  with one cusp. It is a 2-variable polynomial which describes the dependence of the eigenvalues of a meridian and longitude under any representation of  $\pi_1(W)$  into  $\mathrm{SL}(2, \mathbb{C})$ . The  $A$ -polynomial plays a key role in two problems:

- the deformation of the hyperbolic structure of  $W$ ,
- the problem of exceptional (i.e., non-hyperbolic) fillings of  $W$ .

Knowledge of the  $A$ -polynomial (and often, of its Newton polygon) is translated directly into information about the above problems, and vice-versa. In particular, as demonstrated by Boyer and Zhang [BZ01], the Newton polygon is dual to the fundamental polygon of the Culler-Shalen seminorm [CGLS87] and, therefore, can be used to classify cyclic and finite exceptional surgeries.

In [Gar10], the first author observed a pattern in the behavior of the  $A$ -polynomial (and its Newton polygon) of a 1-parameter family of 3-manifolds obtained by fillings of a 2-cusped manifold. To state the pattern, we need to introduce some notation. Let  $K = \mathbb{Q}(x_1, \dots, x_r)$  denote the field of rational functions in  $r$  variables  $x_1, \dots, x_r$ .

**Definition 1.1.** We say that a sequence of rational functions  $R_n \in K$  (defined for all integers  $n$ ) is *holonomic* if it satisfies a linear recursion with constant coefficients. In other words, there exists a natural number  $d$  and  $c_k \in K$  for  $k = 0, \dots, d$  with  $c_d c_0 \neq 0$  such that for all integers  $n$  we have:

$$(1) \quad \sum_{k=0}^d c_k R_{n+k} = 0$$

Depending on the circumstances, one can restrict attention to sequences indexed by the natural numbers (rather than the integers).

Consider a hyperbolic manifold  $W$  with two cusps  $C_1$  and  $C_2$ . Let  $(\mu_i, \lambda_i)$  for  $i = 1, 2$  be pairs of meridian-longitude curves, and let  $W_n$  denote the result of  $-1/n$  filling on  $C_2$ . Let  $A_n(M_1, L_1)$  denote the  $A$ -polynomial of  $W_n$  with the meridian-longitude pair inherited from  $W$ .

**Theorem 1.1.** [Gar10] *With the above conventions, there exists a holonomic sequence  $R_n(M_1, L_1) \in \mathbb{Q}(M_1, L_1)$  such that for all but finitely many integers  $n$ ,  $A_n(M_1, L_1)$  divides the numerator of  $R_n(M_1, L_1)$ . In addition, a recursion for  $R_n$  can be computed explicitly via elimination, from an ideal triangulation of  $W$ .*

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**1.2. The Newton polytope of a holonomic sequence.** Theorem 1.1 motivates us to study the Newton polytope of a holonomic sequence of Laurent polynomials. To state our result, we need some definitions. Recall that the *Newton polytope* of a Laurent polynomial in  $n$  variables  $x_1, \dots, x_n$  is the convex hull of the points whose coordinates are the exponents of its monomials. Recall that a *quasi-polynomial* is a function  $p : \mathbb{N} \rightarrow \mathbb{Q}$  of the form  $p(n) = \sum_{k=0}^d c_k(n)n^k$  where  $c_k : \mathbb{N} \rightarrow \mathbb{Q}$  are periodic functions. When  $c_d \neq 0$ , we call  $d$  the *degree* of  $p(n)$ . We will call quasi-polynomials of degree at most one (resp. two) *quasi-linear* (resp. *quasi-quadratic*). Quasi-polynomials appear in lattice point counting problems (see [Ehr62, CW10]), in the Slope Conjecture in quantum topology (see [Gar11b]), in enumerative combinatorics (see [Gar11a]) and also in the  $A$ -polynomial of filling families of 3-manifolds (see [Gar10]).

**Definition 1.2.** We say that a sequence  $N_n$  of polytopes is linear (resp. quasi-linear) if the coordinates of the vertices of  $N_n$  are polynomials (resp. quasi-polynomials) in  $n$  of degree at most one. Likewise, we say that a sequence  $N_n$  of polytopes is quadratic (resp. quasi-quadratic) if the coordinates of the vertices of  $N_n$  are polynomials (resp. quasi-polynomials) of degree at most two.

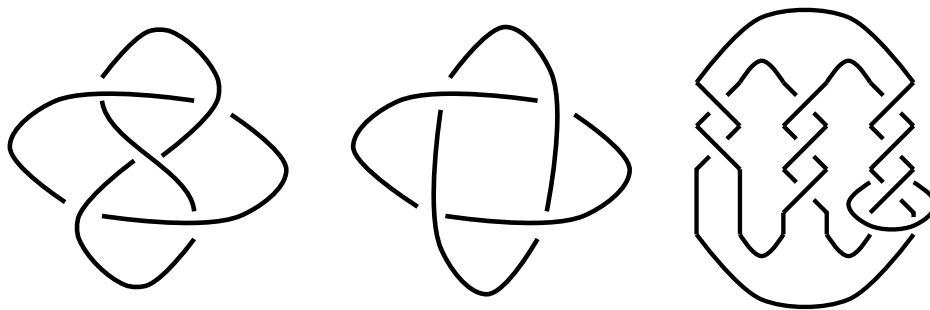
**Theorem 1.2.** [Gar10] *Let  $N_n$  be the Newton polytope of a holonomic sequence  $R_n \in \mathbb{Q}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ . Then, for all but finitely many integers  $n$ ,  $N_n$  is quasi-linear.*

**1.3. Do favorable links exist?** Theorems 1.1 and 1.2 are general, but in favorable circumstances more is true. Namely, consider a family of knot complements  $K_n$ , obtained by  $-1/n$  filling on a cusp of a 2-component hyperbolic link  $J$ . Let  $f$  denote the linking number of the two components of  $J$ , and let  $A_n$  denote the  $A$ -polynomial of  $K_n$  with respect to its canonical meridian and longitude  $(M, L)$ . By definition,  $A_n$  contains all components of irreducible representations, but *not* the component  $L - 1$  of abelian representations.

**Definition 1.3.** We say that  $J$ , a 2-component link in 3-space, with linking number  $f$  is *favorable* if  $A_n(M, LM^{-f^2n}) \in \mathbb{Q}[M^{\pm 1}, L^{\pm 1}]$  is holonomic.

The shift of coordinates,  $LM^{-f^2n}$ , above is due to the canonical meridian-longitude pair of  $K_n$  differing from the corresponding pair for the unfilled component of  $J$  as a result of the nonzero linking number. Theorem 1.2 combined with the above shift implies that, for a favorable link, the Newton polygon of  $K_n$  is quasi-quadratic.

Hoste-Shanahan studied the first examples of a favorable link, the *Whitehead link* and its *half-twisted* version (see Figure 1), and consequently gave an explicit recursion relation for the 1-parameter families of  $A$ -polynomials of twist knots  $K_{2,n}$  and  $K_{3,n}$  respectively; see [HS04].



**Figure 1.** The Whitehead link on the left, the half-twisted Whitehead link in the middle and our seed link  $J$  at right.

The goal of our paper is to give another example of a favorable link  $J$  (see Figure 1), whose 1-parameter filling gives rise to the family of  $(-2, 3, 3 + 2n)$  *pretzel knots*. Our paper is a concrete illustration of the general Theorems 1.1 and 1.2 above. Aside from this, the 1-parameter family of knots  $K_n$ , where  $K_n$  is the  $(-2, 3, 3 + 2n)$  pretzel knot, is well-studied in hyperbolic geometry (where  $K_n$  and the mirror of  $K_{-n}$  are pairs of geometrically similar knots; see [BH96, MM08]), in exceptional Dehn surgery (where for instance

$K_2 = (-2, 3, 7)$  has three Lens space fillings  $1/0$ ,  $18/1$  and  $19/1$ ; see [CGLS87]) and in Quantum Topology (where  $K_n$  and the mirror of  $K_{-n}$  have different Kashaev invariant, equal volume, and different subleading corrections to the volume, see [GZ]).

The success of Theorems 1.3 and 1.4 below hinges on two independent results of Tamura-Yokota and the second author [TY04, Mat02], and an additional lucky coincidence. Tamura-Yokota compute an explicit recursion relation, as in Theorem 1.3, by elimination, using the gluing equations of the decomposition of the complement of  $J$  into six ideal tetrahedra; see [TY04]. The second author computes the Newton polygon  $N_n$  of the  $A$ -polynomial of the family  $K_n$  of pretzel knots; see [Mat02]. This part is considerably more difficult, and requires:

- (a) The set of boundary slopes of  $K_n$ , which are available by applying the Hatcher-Oertel algorithm [HO89, Dun01] to the 1-parameter family  $K_n$  of Montesinos knots. The four slopes given by the algorithm are candidates for the slopes of the sides of  $N_n$ . Similarly, the fundamental polygon of the Culler-Shalen seminorm of  $K_n$  has vertices in rays which are the multiples of the slopes of  $N_n$ . Taking advantage of the duality of the fundamental polygon and Newton polygon, in order to describe  $N_n$  it is enough to determine the vertices of the Culler-Shalen polygon.
- (b) Use of the exceptional  $1/0$  filling and two fortunate exceptional Seifert fillings of  $K_n$  with slopes  $4n + 10$  and  $4n + 11$  to determine exactly the vertices of the Culler-Shalen polygon and consequently  $N_n$ . In particular, the boundary slope 0 is not a side of  $N_n$  (unless  $n = -3$ ) and the Newton polygon is a hexagon for all hyperbolic  $K_n$ .

Given the work of [TY04] and [Mat02], if one is lucky enough to match  $N_n$  of [Mat02] with the Newton polygon of the solution of the recursion relation of [TY04] (and also match a leading coefficient), then Theorem 1.3 below follows; i.e.,  $J$  is a favorable link.

**1.4. Our results for the pretzel knots  $K_n$ .** Let  $A_n(M, L)$  denote the  $A$ -polynomial of the pretzel knot  $K_n$ , using the canonical meridian-longitude coordinates. Consider the sequences of Laurent polynomials  $P_n(M, L)$  and  $Q_n(M, L)$  defined by:

$$(2) \quad P_n(M, L) = A_n(M, LM^{-4n})$$

for  $n > 1$  and

$$(3) \quad Q_n(M, L) = A_n(M, LM^{-4n})M^{-4(3n^2+11n+4)}$$

for  $n < -2$  and  $Q_{-2}(M, L) = A_{-2}(M, LM^{-8})M^{-20}$ . In the remaining cases  $n = -1, 0, 1$ , the knot  $K_n$  is not hyperbolic (it is the torus knot  $5_1$ ,  $8_{19}$  and  $10_{124}$  respectively), and one expects exceptional behavior. This is reflected in the fact that  $P_n$  for  $n = 0, 1$  and  $Q_n$  for  $n = -1, 0$  can be defined to be suitable rational functions (rather than polynomials) of  $M, L$ . Let  $NP_n$  and  $NQ_n$  denote the Newton polygons of  $P_n$  and  $Q_n$  respectively.

**Theorem 1.3.** (a)  $P_n$  and  $Q_n$  satisfy linear recursion relations

$$(4) \quad \sum_{k=0}^4 c_k P_{n+k} = 0, \quad n \geq 0$$

and

$$(5) \quad \sum_{k=0}^4 c_k Q_{n-k} = 0, \quad n \leq 0$$

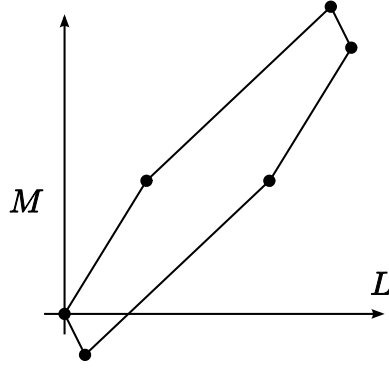
where the coefficients  $c_k$  and the initial conditions  $P_n$  for  $n = 0, \dots, 3$  and  $Q_n$  for  $n = -3, \dots, 0$  are given in Appendix A.

(b) In  $(L, M)$  coordinates,  $NP_n$  and  $NQ_n$  are hexagons with vertices

$$(6) \quad \{[0, 0], [1, -4n + 16], [n - 1, 12n - 12], [2n + 1, 16n + 18], [3n - 1, 32n - 10], [3n, 28n + 6]\}$$

for  $P_n$  with  $n > 1$  and

$$(7) \quad \{[0, 4n + 28], [1, 38], [-n, -12n + 26], [-2n - 3, -16n - 4], [-3n - 4, -28n - 16], [-3n - 3, -32n - 6]\}$$



**Figure 2.** The Newton polygon  $NP_n$ .

for  $Q_n$  with  $n < -1$ .

**Remark 1.4.** We can give a single recursion relation valid for  $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$  as follows. Define

$$(8) \quad R_n(M, L) = A_n(M, LM^{-4n})b^{|n|}\epsilon_n(M),$$

where

$$(9) \quad b = \frac{1}{LM^8(1-M^2)(1+LM^{10})} \quad c = \frac{L^3M^{12}(1-M^2)^3}{(1+LM^{10})^3} \quad \epsilon_n(M) = \begin{cases} 1 & \text{if } n > 1 \\ cM^{-4(3+n)(2+3n)} & \text{if } n < -2 \\ cM^{-28} & \text{if } n = -2 \end{cases}$$

Then,  $R_n$  satisfies the palindromic fourth order linear recursion

$$(10) \quad \sum_{k=0}^4 \gamma_k R_{n+k} = 0$$

where the coefficients  $\gamma_k$  and the initial conditions  $R_n$  for  $n = 0, \dots, 3$  are given in Appendix B. Moreover,  $R_n$  is related to  $P_n$  and  $Q_n$  by:

$$(11) \quad R_n = \begin{cases} P_n b^{|n|} & \text{if } n \geq 0 \\ Q_n b^{|n|} c M^{-8} & \text{if } n \leq 0 \end{cases}$$

**Remark 1.5.** The computation of the Culler-Shalen seminorm of the pretzel knots  $K_n$  has an additional application, namely it determines the number of components (containing the character of an irreducible representation) of the  $SL(2, \mathbb{C})$  character variety of the knot, and consequently the number of factors of its  $A$ -polynomial. In the case of  $K_n$ , (after translating the results of [Mat02] for the pretzel knots  $(-2, 3, n)$  to the pretzel knots  $(-2, 3, 3+2n)$ ) it was shown by the second author [Mat02, Theorem 1.6] that the character variety of  $K_n$  has one (resp. two) components when 3 does not divide  $n$  (resp. divides  $n$ ). The non-geometric factor of  $A_n$  is given by

$$\begin{cases} 1 - LM^{4(n+3)} & n \geq 3 \\ L - M^{-4(n+3)} & n \leq -3 \end{cases}$$

for  $n \neq 0$  a multiple of 3.

Since the  $A$ -polynomial has even powers of  $M$ , we can define the  $B$ -polynomial by

$$B(M^2, L) = A(M, L).$$

Our next result relates the  $A$ -polynomials of the geometrically similar pair  $(K_n, -K_{-n})$  by an explicit  $GL(2, \mathbb{Z})$  transformation.

**Theorem 1.4.** *For  $n > 1$  we have:*

$$(12) \quad B_{-n}(M, LM^{2n-5}) = (-L)^n M^{3(2n^2-7n+7)} B_n(-L^{-1}, L^{2n+5} M^{-1}) \eta_n$$

where  $\eta_n = 1$  (resp.  $M^{22}$ ) when  $n > 2$  (resp.  $n = 2$ ).

## 2. PROOFS

**2.1. The equivalence of Theorem 1.3 and Remark 1.4.** In this subsection we will show the equivalence of Theorem 1.3 and Remark 1.4. Let  $\gamma_k = c_k/b^k$  for  $k = 0, \dots, 4$  where  $b$  is given by (9). It is easy to see that the  $\gamma_k$  are given explicitly by Appendix B, and moreover, they are palindromic. Since  $R_n = P_n b^n$  for  $n = 0, \dots, 3$  it follows that  $R_n$  and  $P_n b^n$  satisfy the same recursion relation (10) for  $n \geq 0$  with the same initial conditions. It follows that  $R_n = P_n b^n$  for  $n \geq 0$ .

Solving (10) backwards, we can check by an explicit calculation that  $R_n = Q_n b^{|n|} c M^{-8}$  for  $n = -3, \dots, 0$  where  $b$  and  $c$  are given by (9). Moreover,  $R_n$  and  $Q_n b^{|n|} c M^{-8}$  satisfy the same recursion relation (10) for  $n < 0$ . It follows that  $R_n = Q_n b^{|n|} c M^{-8}$  for  $n < 0$ . This concludes the proof of Equations (10) and (11).

**2.2. Proof of Theorem 1.3.** Let us consider first the case of  $n \geq 0$ , and denote by  $P'_n$  for  $n \geq 0$  the unique solution to the linear recursion relation (4) with the initial conditions as in Theorem 1.3. Let  $R'_n = P'_n b^n$  be defined according to Equation (11) for  $n \geq 0$ .

Remark 1.4 implies that  $R'_n$  satisfies the recursion relation of [TY04, Thm.1]. It follows by [TY04, Thm.1] that  $A_n(M, LM^{-4n})$  divides  $P'_n(M, L)$  when  $n > 1$ .

Next, we claim that the Newton polygon  $\text{NP}'_n$  of  $P'_n(M, L)$  is given by (6). This can be verified easily by induction on  $n$ .

Next, in [Mat02, p.1286], the second author computes the Newton polygon  $N_n$  of the  $A_n(M, L)$ . It is a hexagon given in  $(L, M)$  coordinates by

$$\begin{aligned} & \{\{0, 0\}, \{1, 16\}, \{n-1, 4(n^2+2n-3)\}, \{2n+1, 2(4n^2+10n+9)\}, \\ & \{3n-1, 2(6n^2+14n-5)\}, \{3n, 2(6n^2+14n+3)\}\} \end{aligned}$$

when  $n > 1$ ,

$$\begin{aligned} & \{\{-3n-4, 0\}, \{-3(1+n), 10\}, \{-3-2n, 4(3+4n+n^2)\}, \\ & \{-n, 2(4n^2+16n+21)\}, \{0, 4(3n^2+12n+11)\}, \{1, 6(2n^2+8n+9)\}\} \end{aligned}$$

when  $n < -2$  and

$$\{\{0, 0\}, \{1, 0\}, \{2, 4\}, \{1, 10\}, \{2, 14\}, \{3, 14\}\}$$

when  $n = -2$ . Notice that the above 1-parameter families of Newton polygons are quadratic. It follows by explicit calculation that the Newton polygon of  $A_n(M, LM^{-4n})$  is quadratic and exactly agrees with  $\text{NP}'_n$  for all  $n > 1$ .

The above discussion implies that  $P_n(M, L)$  is a rational multiple of  $A_n(M, LM^{-4n})$ . Since their leading coefficients (with respect to  $L$ ) agree, they are equal. This proves Theorem 1.3 for  $n > 1$ . The case of  $n < -1$  is similar.  $\square$

**2.3. Proof of Theorem 1.4.** Using Equations (2) and (3), convert Equation (12) into

$$(13) \quad Q_{-n}(\sqrt{M}, L/M^5) = (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M).$$

Note that, under the substitution  $(M, L) \mapsto (i/\sqrt{L}, L^{2n+5}/M)$ ,  $LM^{4n}$  becomes  $L^5/M$ . Similarly,  $LM^{-4n}$  becomes  $L/M^5$  under the substitution  $(M, L) \mapsto (\sqrt{M}, LM^{2n-5})$ .

It is straightforward to verify equation (13) for  $n = 2, 3, 4, 5$ . For  $n \geq 6$ , we use induction. Let  $c_k^-$  denote the result of applying the substitutions  $(M, L) \mapsto (\sqrt{M}, L/M^5)$  to the  $c_k$  coefficients in the recursions (4) and (5). For example,

$$c_0^- = \frac{L^4(1+L)^4(1-M)^4}{M^2}.$$

Similarly, define  $c_k^+$  to be the result of the substitution  $(M, L) \mapsto (i/\sqrt{L}, L^5/M)$  to  $c_k$ . It is easy to verify that for  $k = 0, 1, 2, 3$ ,

$$\frac{c_k^-}{c_4^-}(-LM)^{k-4} = \frac{c_k^+}{c_4^+}.$$

Then,

$$\begin{aligned} Q_{-n}(\sqrt{M}, L/M^5) &= -\frac{1}{c_4^-} \sum_{k=0}^3 c_k^- Q_{-n+4-k}(\sqrt{M}, L/M^5) \\ &= -\frac{1}{c_4^-} \sum_{k=0}^3 c_k^- (-L)^{n-4+k} M^{n-4+k+13} P_{n-4+k}(i\sqrt{L}, L^5/M) \\ &= -(-L)^n M^{n+13} \sum_{k=0}^3 \frac{c_k^-}{c_4^-} (-LM)^{k-4} P_{n-4+k}(i\sqrt{L}, L^5/M) \\ &= -(-L)^n M^{n+13} \sum_{k=0}^3 \frac{c_k^+}{c_4^+} P_{n-4+k}(i\sqrt{L}, L^5/M) \\ &= (-L)^n M^{n+13} P_n(i\sqrt{L}, L^5/M). \end{aligned}$$

By induction, equation (13) holds for all  $n > 1$  proving Theorem 1.4.  $\square$

#### APPENDIX A. THE COEFFICIENTS $c_k$ AND THE INITIAL CONDITIONS FOR $P_n$ AND $Q_n$

$$\begin{aligned} c_4 &= M^4 \\ c_3 &= 1 + M^4 + 2LM^{12} + LM^{14} - LM^{16} + L^2M^{20} - L^2M^{22} - 2L^2M^{24} - L^3M^{32} - L^3M^{36} \\ c_2 &= (-1 + LM^{12})(-1 - 2LM^{10} - 3LM^{12} + 2LM^{14} - L^2M^{16} + 2L^2M^{18} - 4L^2M^{20} - 2L^2M^{22} + 3L^2M^{24} \\ &\quad - 3L^3M^{28} + 2L^3M^{30} + 4L^3M^{32} - 2L^3M^{34} + L^3M^{36} - 2L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} + L^5M^{52}) \\ c_1 &= -L^2(-1 + M)^2M^{16}(1 + M)^2(1 + LM^{10})^2(-1 - M^4 - 2LM^{12} - LM^{14} + LM^{16} - L^2M^{20} + L^2M^{22} \\ &\quad + 2L^2M^{24} + L^3M^{32} + L^3M^{36}) \\ c_0 &= L^4(-1 + M)^4M^{36}(1 + M)^4(1 + LM^{10})^4 \end{aligned}$$

$$\begin{aligned} P_0 &= \frac{(-1 + LM^{12})(1 + LM^{12})^2}{(1 + LM^{10})^3} \\ P_1 &= \frac{(-1 + LM^{11})^2(1 + LM^{11})^2}{1 + LM^{10}} \\ P_2 &= -1 + LM^8 - 2LM^{10} + LM^{12} + 2L^2M^{20} + L^2M^{22} - L^4M^{40} - 2L^4M^{42} - L^5M^{50} + 2L^5M^{52} - L^5M^{54} \\ &\quad + L^6M^{62} \\ P_3 &= (-1 + LM^{12})(-1 + LM^4 - LM^6 + 2LM^8 - 5LM^{10} + LM^{12} + 5L^2M^{16} - 4L^2M^{18} + L^2M^{22} + L^3M^{26} \\ &\quad + 3L^3M^{30} + 2L^3M^{32} - 2L^4M^{36} - 3L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} - 2L^5M^{46} - 3L^5M^{48} - L^5M^{52} \\ &\quad - L^6M^{56} + 4L^6M^{60} - 5L^6M^{62} - L^7M^{66} + 5L^7M^{68} - 2L^7M^{70} + L^7M^{72} - L^7M^{74} + L^8M^{78}) \end{aligned}$$

$$\begin{aligned}
Q_0 &= -\frac{(-1 + LM^{12})(1 + LM^{12})^2}{L^3(-1 + M)^3 M^4(1 + M)^3} \\
Q_{-1} &= -\frac{M^{12}(1 + LM^{14})^2}{L(-1 + M)(1 + M)} \\
Q_{-2} &= M^{20}(1 - LM^8 + 2LM^{10} + 2LM^{12} - LM^{16} + LM^{18} + L^2M^{20} - L^2M^{22} + 2L^2M^{26} + 2L^2M^{28} \\
&\quad - L^2M^{30} + L^3M^{38}) \\
Q_{-3} &= M^{16}(-1 + LM^{12})(1 + LM^{10} + 5LM^{12} - LM^{14} - 2LM^{16} + 2LM^{18} - LM^{20} + 2L^2M^{20} + LM^{22} \\
&\quad + 4L^2M^{22} + 3L^2M^{26} - 3L^2M^{28} - L^3M^{28} + 5L^3M^{30} + 5L^2M^{32} - L^2M^{34} - 3L^3M^{34} + 3L^3M^{36} \\
&\quad + 4L^3M^{40} + L^4M^{40} + 2L^3M^{42} - L^4M^{42} + 2L^4M^{44} - 2L^4M^{46} - L^4M^{48} + 5L^4M^{50} + L^4M^{52} + L^5M^{62})
\end{aligned}$$

APPENDIX B. THE COEFFICIENTS  $\gamma_k$  AND THE INITIAL CONDITIONS FOR  $R_n$ 

$$\begin{aligned}
\gamma_4 &= L^4(-1 + M)^4 M^{36}(1 + M)^4(1 + LM^{10})^4 \\
\gamma_3 &= L^3(-1 + M)^3 M^{24}(1 + M)^3(1 + LM^{10})^3(-1 - M^4 - 2LM^{12} - LM^{14} + LM^{16} - L^2M^{20} + L^2M^{22} \\
&\quad + 2L^2M^{24} + L^3M^{32} + L^3M^{36}) \\
\gamma_2 &= L^2(-1 + M)^2 M^{16}(1 + M)^2(1 + LM^{10})^2(-1 + LM^{12})(-1 - 2LM^{10} - 3LM^{12} + 2LM^{14} - L^2M^{16} \\
&\quad + 2L^2M^{18} - 4L^2M^{20} - 2L^2M^{22} + 3L^2M^{24} - 3L^3M^{28} + 2L^3M^{30} + 4L^3M^{32} - 2L^3M^{34} + L^3M^{36} \\
&\quad - 2L^4M^{38} + 3L^4M^{40} + 2L^4M^{42} + L^5M^{52}) \\
\gamma_1 &= \gamma_3 \\
\gamma_0 &= \gamma_4
\end{aligned}$$

Let  $P_n$  for  $n = 0, \dots, 3$  be as in Appendix A. Then,

$$(14) \quad R_n = P_n b^n$$

for  $n = 0, \dots, 3$  where  $b$  is given by Equation (9).

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